J. Math. Anal. Appl. 381 (2011) 957–962

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

Note on ground states of a nonlinear Schrödinger system

Jinyong Chang

Department of Mathematics, Changzhi University, Shanxi 046011, PR China

ARTICLE INFO

Article history:
Received 11 December 2010
Available online 18 February 2011
Submitted by Steven G. Krantz

Keywords:
Schrödinger system
Nontrivial ground state
Morse index

ABSTRACT

We give a sufficient condition for the existence of positive radial ground states of the time-independent Schrödinger system

$$\begin{cases} -\Delta u_1 + \lambda u_1 = \mu_1 u_1^3 + \beta_{12} u_2^2 u_1 + \beta_{13} u_3^2 u_1, & \text{in } \mathbb{R}^n, \\ -\Delta u_2 + \lambda u_2 = \beta_{12} u_1^2 u_2 + \mu_2 u_2^3 + \beta_{23} u_3^2 u_2, & \text{in } \mathbb{R}^n, \\ -\Delta u_3 + \lambda u_3 = \beta_{13} u_1^2 u_3 + \beta_{23} u_2^2 u_3 + \mu_3 u_3^3, & \text{in } \mathbb{R}^n, \\ u_1(x) \to 0, \quad u_2(x) \to 0, \quad u_3(x) \to 0, & \text{as } |x| \to \infty, \end{cases}$$

where $n = 1, 2, 3$, $\lambda > 0$, $\mu_j > 0$ and $\beta_{ij} > 0$ $(i < j)$ for $i, j = 1, 2, 3$. And in some special cases, our conditions are also necessary.

In this paper, we are concerned with solitary wave solutions of time-dependent nonlinear Schrödinger equations given by

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta_{12}|\Phi_2|^2\Phi_1 + \beta_{13}|\Phi_3|^2\Phi_1, & \text{in } \mathbb{R}^n, \\ -i\dfrac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \beta_{21}|\Phi_1|^2\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta_{23}|\Phi_3|^2\Phi_2, & \text{in } \mathbb{R}^n, \\ -i\dfrac{\partial}{\partial t}\Phi_2 = \Delta\Phi_3 + \beta_{31}|\Phi_1|^2\Phi_3 + \beta_{32}|\Phi_2|^2\Phi_3 + \mu_3|\Phi_3|^2\Phi_3, & \text{in } \mathbb{R}^n, \\ \Phi_1(x,t) \to 0, \quad \Phi_2(x,t) \to 0, \quad \Phi_3(x,t) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1}$$

where $n = 1, 2, 3$, $\mu_j > 0$, $\beta_{ij} = \beta_{ji} > 0$ for $i \neq j$ and $i, j = 1, 2, 3$.

The model (1) has applications in many physical problems, especially in nonlinear optics. An application of (1) comes from [1,16], the solution Φ_j denotes the jth component of the beam in Kerr-like photorefractive media. The constant μ_j is for self-focusing in the jth component of the beam. The coupling constant β_{ij} is the interaction between the ith and the jth component of the beam. The interaction is attractive if $\beta_{ij} > 0$, and repulsive if $\beta_{ij} < 0$. For more references we refer the reader to [1,4,6–11].

A solitary wave of (1) is a solution with $\Phi_j(x,t) = e^{i\lambda_j t}u_j(x)$, $j = 1, 2, 3$. This ansatz leads to the elliptic system

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta_{12} u_2^2 u_1 + \beta_{13} u_3^2 u_1, & \text{in } \mathbb{R}^n, \\ -\Delta u_2 + \lambda_2 u_2 = \beta_{12} u_1^2 u_2 + \mu_2 u_2^3 + \beta_{23} u_3^2 u_2, & \text{in } \mathbb{R}^n, \\ -\Delta u_3 + \lambda_3 u_3 = \beta_{13} u_1^2 u_3 + \beta_{23} u_2^2 u_3 + \mu_3 u_3^3, & \text{in } \mathbb{R}^n, \\ u_1(x) \to 0, \quad u_2(x) \to 0, \quad u_3(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{2}$$

E-mail address: jinyong_chang@yahoo.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.02.043

In the last several years there have been intensive work on the existence, multiplicity and qualitative property of ground and bound state nontrivial solutions for systems like (2) (see [2,3,13–15,17] for instance). Here and below by a nontrivial solution of (2), we mean a solution $\vec{u}=(u_1,u_2,u_3)$ with each component u_j being nonzero. It is an important feature of the study for these type of systems that one needs to distinguish nontrivial solutions from semitrivial solutions (solutions with one or two components being zero). In addition, a solution $\vec{u}=(u_1,u_2,u_3)$ of (2) is called a positive solution if $u_1>0$, $u_2>0$ and $u_3>0$, and a semipositive solution if $u_1\geqslant 0$, $u_2\geqslant 0$ and $u_3\geqslant 0$ and if at least one of them is not zero. A solution is said to be a ground state if it has the least energy among all semitrivial and nontrivial solutions.

The energy functional corresponding to (2) is defined by

$$E(\vec{u})=\frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u_1|^2+\lambda_1 u_1^2+|\nabla u_2|^2+\lambda_2 u_2^2+|\nabla u_3|^2+\lambda_3 u_3^2\right)$$
$$-\frac{1}{4}\int_{\mathbb{R}^n}\left(\mu_1 u_1^4+\mu_2 u_2^4+\mu_3 u_3^4+2\beta_{12}u_1^2u_2^2+2\beta_{13}u_1^2u_3^2+2\beta_{23}u_2^2u_3^2\right)$$

for $\vec{u}=(u_1,u_2,u_3)\in (H^1(\mathbb{R}^n))^3$. Hence, solutions of (2) correspond to critical points of it.

Let $H_r^1(\mathbb{R}^n)$ consist of all radial functions in $H^1(\mathbb{R}^n)$. Denote $X=(H^1(\mathbb{R}^n))^3$ and $X_r=(H_r^1(\mathbb{R}^n))^3$. It was proved in [3] that (2) has a semipositive radial ground state which is of mountain pass type and has Morse index 1 when considered as the critical point of E on X and on X_r. But it seems to be an open problem that on what conditions (2) has a positive ground state. There is a partial answer from the recent work [15] by Liu and Wang. In [15], they provide a sufficient condition for the existence of a nontrivial ground state solution for a general N-system by comparing the energies.

Our paper concerns only about 3-system like (2). We also give a sufficient condition for (2) by computing the Morse indices of nontrivial and semitrivial solutions and it seems to be better than that of [15] in a special case. Nevertheless, we do not give the comparison in general case as the assumptions in [15] involve heavy notations and are described with quite complex inequalities, which are not easy to verify.

The argument of this paper is motivated by those in [3,5]. In [5], by comparing the Morse indices of nontrivial solutions with semitrivial solutions, the authors obtained the existence of nontrivial ground state for 2-system like

$$\begin{cases}-\Delta u_1+\lambda_1 u_1=\mu_1 u_1^3+\beta_{12}u_2^2u_1, & \text{in } \mathbb{R}^n,\\ -\Delta u_2+\lambda_2 u_2=\beta_{12}u_1^2u_2+\mu_2 u_2^3, & \text{in } \mathbb{R}^n,\\ u_1(x)\to 0,\quad u_2(x)\to 0, & \text{as } |x|\to\infty.\end{cases}\tag{3}$$

To search for all the semitrivial solutions of (3), we need to know the uniqueness of positive solutions of

$$-\Delta u_j+\lambda_j u_j=\mu_j u_j^3\quad \text{in } \mathbb{R}^n.$$

However, this has been proved in [12]. We denote them by $w_j(x)=\sqrt{\frac{\lambda_j}{\mu_j}}\,w(\sqrt{\lambda_j}x)$. Similarly, to search for all the semitrivial solutions of (2), we also need to know the uniqueness of positive solutions of (3). And the result was obtained in [18] recently.

And now we state our results in the following:

Theorem 1. *Assume $\lambda_1=\lambda_2=\lambda_3=\lambda$. If*

$$\beta_{ij}>\max\{\mu_i,\mu_j\}\quad\text{and}\quad \beta_{ij}^2-\mu_i\mu_j<\beta_{ik}(\beta_{ij}-\mu_j)+\beta_{jk}(\beta_{ij}-\mu_i),\tag{4}$$

for $i,j,k=1,2,3$, $i<j$ and $i\neq k$, $j\neq k$, then (2) has a nontrivial radial ground state.

Remark 2. (a) If $\beta_{12}=\beta_{13}=\beta_{23}=\beta$, then (4) reduces to $\beta>\max\{\mu_1,\mu_2,\mu_3\}$. And it seems to be better than the results of [15], which need $\beta>6\max\{\mu_1,\mu_2,\mu_3\}$ (see [Corollary 2.3] therein).

(b) The argument provided here may be generalized to systems consisting of N equations, providing we know the uniqueness and concrete forms of the positive solutions of $N-1$ equations.

(c) It seems that (4) is only a sufficient condition for (2) to have a nontrivial radial ground state when $\lambda_1=\lambda_2=\lambda_3=\lambda$. And we do not know whether it is also necessary. But when $\beta_{12}=\beta_{13}=\beta_{23}$, to some extent, it seems to be necessary. The reason is that we can follow the idea of [3] to show the following theorems.

Theorem 3. *Let $n=1$, λ be as in Theorem 1 and $\beta_{12}=\beta_{13}=\beta_{23}=\beta$. Then (2) has a positive ground state if and only if $\beta>\max\{\mu_1,\mu_2,\mu_3\}$ or $\beta=\mu_1=\mu_2=\mu_3$.*

And in the higher-dimensional case, we have the following result.

Theorem 4. *Let $n = 2, 3$, λ be as in Theorem* 1, $\min\{\mu_1, \mu_2, \mu_3\} < \max\{\mu_1, \mu_2, \mu_3\}$ *and* $\beta_{12} = \beta_{13} = \beta_{23} = \beta$. *Then* (2) *has a positive ground state if and only if* $\beta > \max\{\mu_1, \mu_2, \mu_3\}$.

We first give the proofs of Theorems 3 and 4 in the following:

Proof of Theorem 3. Without loss of generality we may assume $\mu_1 = \min\{\mu_1, \mu_2, \mu_3\}$, $\mu_2 = \max\{\mu_1, \mu_2, \mu_3\}$.

If $\beta > \mu_2$, then (2) has a positive ground state according to Theorem 1.

For $0 \leqslant \beta < \mu_1$, we claim that any positive solution $\vec{u} = (u_1, u_2, u_3)$ has Morse index at least 2. Let $\vec{\phi} = (t_1u_1, t_2u_2, t_3u_3) \in X$. A direct computation shows

$$E''(\vec{u})[\vec{\phi}, \vec{\phi}] = -2\mu_1 t_1^2 \int_{\mathbb{R}^n} u_1^4 - 2\mu_2 t_2^2 \int_{\mathbb{R}^n} u_2^4 - 2\mu_3 t_3^2 \int_{\mathbb{R}^n} u_3^4 - 4\beta t_1 t_2 \int_{\mathbb{R}^n} u_1^2 u_2^2 - 4\beta t_1 t_3 \int_{\mathbb{R}^n} u_1^2 u_3^2 - 4\beta t_2 t_3 \int_{\mathbb{R}^n} u_2^2 u_3^2.$$

Using $\beta < \mu_1$, we see the above is negative for all $(t_1, t_2, 0) \neq (0, 0, 0)$. This shows that the ground state must have one component zero by Theorem 1.1 of [3].

Finally, if $\mu_1 \leqslant \beta \leqslant \mu_2$, we assume there is a positive solution $\vec{u} = (u_1, u_2, u_3)$ of system (2). Multiplying the first by u_2 and the second equation by u_1 and then integrating by parts on $\mathbb{R}^n$ and subtracting one from the other, we obtain

$$0 = \int_{\mathbb{R}^n} (\mu_1 - \beta) u_1^3 u_2 + \int_{\mathbb{R}^n} (\beta - \mu_2) u_2^3 u_1.$$

Therefore, (2) admits no positive solution unless $\beta = \mu_1 = \mu_2$.

If $\beta = \mu_1 = \mu_2$, then all the semipositive solutions that are not positive are $U_1^* = (w_1, 0, 0)$, $U_2^* = (0, w_1, 0)$, $U_3^* = (0, 0, w_1)$, $U_4^* = (w_1 \cos\theta_1, w_1 \sin\theta_1, 0)$, $U_5^* = (w_1 \cos\theta_2, 0, w_1 \sin\theta_2)$ and $U_6^* = (0, w_1 \cos\theta_3, w_1 \sin\theta_3)$, where $\theta_1, \theta_2, \theta_3 \in (0, \frac{\pi}{2})$, by using Theorem 1.2 of [18]. Note that $U_1^*, U_2^*, \ldots, U_6^*$ embed in a family of solutions $\vec{u}(t_1, t_2, t_3) = (t_1 w_1, t_2 w_1, t_3 w_1)$, $t_1^2 + t_2^2 + t_3^2 = 1$, on which the energy is a constant. We claim this family of solutions are ground state solutions. Fix $\mu_1 = \mu_2 = \mu$, let β be a parameter and denote the responding functional by E_β. From the above arguments we know $U_1^*, U_2^*, \ldots, U_6^*$ are the only ground state solutions for $\beta < \mu$. Thus the least energy for E_μ is the same as for E_β with $\beta < \mu$ because the least energy for E_β is continuous in β. The least energy of E_μ is achieved at $U_1^*, U_2^*, \ldots, U_6^*$, hence at $(t_1 w_1, t_2 w_1, t_3 w_1)$ with $t_1^2 + t_2^2 + t_3^2 = 1$. Thus for $t_1, t_2, t_3 > 0$ these must be positive ground states. □

Proof of Theorem 4. The procedure is similar to that of Theorem 3. □

Proof of Theorem 1. We set $U_1 = (w_1, 0, 0)$, $U_2 = (0, w_2, 0)$, $U_3 = (0, 0, w_3)$. Then they are the only three semipositive radial solutions of (2) with two components being zero.

According to [18], $(u_i(x), u_j(x))$ with u_i and u_j defined by

$$u_i = \sqrt{\frac{\lambda(\beta_{ij} - \mu_j)}{\beta_{ij}^2 - \mu_i\mu_j}} w(\sqrt{\lambda}x) \quad \text{and} \quad u_j = \sqrt{\frac{\lambda(\beta_{ij} - \mu_i)}{\beta_{ij}^2 - \mu_i\mu_j}} w(\sqrt{\lambda}x)$$

is the unique positive radial solution of

$$\begin{cases} -\Delta u_i + \lambda u_i = \mu_i u_i^3 + \beta_{ij} u_j^2 u_i, & \text{in } \mathbb{R}^n, \\ -\Delta u_j + \lambda u_j = \beta_{ij} u_i^2 u_j + \mu_j u_j^3, & \text{in } \mathbb{R}^n, \\ u_i(x) \to 0, \quad u_j(x) \to 0, & \text{as } |x| \to \infty, \end{cases}$$

if $\beta_{ij} > \max\{\mu_i, \mu_j\}$. Therefore, we denote

$$U_4 = (u_0, v_0, 0) = \left(\sqrt{\frac{\lambda(\beta_{12} - \mu_2)}{\beta_{12}^2 - \mu_1\mu_2}} w(\sqrt{\lambda}x), \sqrt{\frac{\lambda(\beta_{12} - \mu_1)}{\beta_{12}^2 - \mu_1\mu_2}} w(\sqrt{\lambda}x), 0\right),$$

$$U_5 = \left(u_0^*, 0, w_0\right) = \left(\sqrt{\frac{\lambda(\beta_{13} - \mu_3)}{\beta_{13}^2 - \mu_1\mu_3}} w(\sqrt{\lambda}x), 0, \sqrt{\frac{\lambda(\beta_{13} - \mu_1)}{\beta_{13}^2 - \mu_1\mu_3}} w(\sqrt{\lambda}x)\right)$$

and

$$U_6 = \left(0, v_0^*, w_0^*\right) = \left(0, \sqrt{\frac{\lambda(\beta_{23} - \mu_3)}{\beta_{23}^2 - \mu_2\mu_3}} w(\sqrt{\lambda}x), \sqrt{\frac{\lambda(\beta_{23} - \mu_2)}{\beta_{23}^2 - \mu_2\mu_3}} w(\sqrt{\lambda}x)\right).$$

Then U_4, U_5 and U_6 are the only three semipositive radial solutions of (2) with only one component being zero.

It has been proved in [3] that (2) has a semipositive radial ground state $\vec{u} \in X_r$ and its Morse index is 1. In order to prove that this ground state solution is positive, it suffices to prove that $U_1, \dots, U_6$ have Morse indices at least 2 in the conditions we give. And we now estimate the Morse indices of them.

(1) For any $(\phi_1, \phi_2, \phi_3) \in (H^1(\mathbb{R}^n))^3$, we have

$$
\begin{aligned}
&E''(U_1)\big[(\phi_1,\phi_2,\phi_3),(\phi_1,\phi_2,\phi_3)\big]\\
&\quad = \int_{\mathbb{R}^n} \big(|\nabla\phi_1|^2 + \lambda\phi_1^2 + |\nabla\phi_2|^2 + \lambda\phi_2^2 + \big|\nabla\phi_3^2\big| + \lambda\phi_3^2\big) - \int_{\mathbb{R}^n} \big(3\mu_1 w_1^2\phi_1^2 + \beta_{12} w_1^2\phi_2^2 + \beta_{13} w_1^2\phi_3^2\big).
\end{aligned}
$$

Thus, for any $t_1, t_2, t_3 \in \mathbb{R}$,

$$
\begin{aligned}
&E''(U_1)\big[(t_1w_1,t_2w_1,t_3w_1),(t_1w_1,t_2w_1,t_3w_1)\big]\\
&\quad = t_1^2\int_{\mathbb{R}^n}\big(|\nabla w_1|^2 + \lambda w_1^2 - 3\mu_1 w_1^4\big) + t_2^2\int_{\mathbb{R}^n}\big(|\nabla w_1|^2 + \lambda w_1^2 - \beta_{12} w_1^4\big) + t_3^2\int_{\mathbb{R}^n}\big(|\nabla w_1|^2 + \lambda w_1^2 - \beta_{13} w_1^4\big)\\
&\quad = t_1^2\int_{\mathbb{R}^n}\big(-2\mu_1 w_1^4\big) + t_2^2\int_{\mathbb{R}^n}(\mu_1-\beta_{12})w_1^4 + t_3^2\int_{\mathbb{R}^n}(\mu_1-\beta_{13})w_1^4.
\end{aligned}
$$

We see that if $\beta_{12} > \mu_1$, then, for any $(t_1, t_2, 0) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$

$$E''(U_1)\big[(t_1w_1,t_2w_1,0),(t_1w_1,t_2w_1,0)\big] < 0.$$

And if $\beta_{13} > \mu_1$, then, for any $(t_1, 0, t_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$

$$E''(U_1)\big[(t_1w_1,0,t_3w_1),(t_1w_1,0,t_3w_1)\big] < 0.$$

Therefore, the Morse index of U_1 is at least 2 if $\max\{\beta_{12}, \beta_{13}\} > \mu_1$. Similarly, the Morse indices of U_2, U_3 are also at least 2 if $\max\{\beta_{12}, \beta_{23}\} > \mu_2$ and $\max\{\beta_{13}, \beta_{23}\} > \mu_3$, respectively.

(2) For any $(\phi_1, \phi_2, \phi_3) \in (H^1(\mathbb{R}^n))^3$, we have

$$
\begin{aligned}
&E''(U_4)\big[(\phi_1,\phi_2,\phi_3),(\phi_1,\phi_2,\phi_3)\big]\\
&\quad = \int_{\mathbb{R}^n} \big(|\nabla\phi_1|^2 + \lambda\phi_1^2 + |\nabla\phi_2|^2 + \lambda\phi_2^2 + \big|\nabla\phi_3^2\big| + \lambda\phi_3^2\big) - \int_{\mathbb{R}^n} \big(3\mu_1 u_0^2\phi_1^2 + 3\mu_2 v_0^2\phi_2^2 + 4\beta_{12}u_0v_0\phi_1\phi_2\big)\\
&\qquad - \int_{\mathbb{R}^n} \big(\beta_{12}u_0^2\phi_2^2 + \beta_{12}v_0^2\phi_1^2 + \beta_{13}u_0^2\phi_3^2 + \beta_{23}v_0^2\phi_3^2\big).
\end{aligned}
$$

Choosing $(\phi_1, \phi_2, \phi_3) = (t_1w_1, t_2w_1, t_3w_1)$, one then obtains

$$
\begin{aligned}
&E''(U_4)\big[(t_1w_1,t_2w_1,t_3w_1),(t_1w_1,t_2w_1,t_3w_1)\big]\\
&\quad = t_1^2\int_{\mathbb{R}^n}\big(|\nabla w_1|^2 + \lambda w_1^2 - 3\mu_1 u_0^2 w_1^2 - \beta_{12} v_0^2 w_1^2\big) + t_2^2\int_{\mathbb{R}^n}\big(|\nabla w_1|^2 + \lambda w_1^2 - 3\mu_2 v_0^2 w_1^2 - \beta_{12} u_0^2 w_1^2\big)\\
&\qquad + t_3^2\int_{\mathbb{R}^n}\big(|\nabla w_1|^2 + \lambda w_1^2 - \beta_{13} u_0^2 w_1^2 - \beta_{23} v_0^2 w_1^2\big) - 4\beta_{12}t_1t_2\int_{\mathbb{R}^n} u_0v_0w_1^2\\
&\quad = t_1^2\bigg[\frac{\lambda^2}{\mu_1} - 3\frac{\lambda^2(\beta_{12}-\mu_2)}{\beta_{12}^2-\mu_1\mu_2} - \frac{\lambda^2\beta_{12}(\beta_{12}-\mu_1)}{\mu_1(\beta_{12}^2-\mu_1\mu_2)}\bigg]\int_{\mathbb{R}^n} w^4(\sqrt{\lambda}x)\\
&\qquad + t_2^2\bigg[\frac{\lambda^2}{\mu_1} - 3\frac{\lambda^2\mu_2(\beta_{12}-\mu_1)}{\mu_1(\beta_{12}^2-\mu_1\mu_2)} - \frac{\lambda^2\beta_{12}(\beta_{12}-\mu_2)}{\mu_1(\beta_{12}^2-\mu_1\mu_2)}\bigg]\int_{\mathbb{R}^n} w^4(\sqrt{\lambda}x)\\
&\qquad + t_3^2\bigg[\frac{\lambda^2}{\mu_1} - \frac{\lambda^2\beta_{13}(\beta_{12}-\mu_2)}{\mu_1(\beta_{12}^2-\mu_1\mu_2)} - \frac{\lambda^2\beta_{23}(\beta_{12}-\mu_1)}{\mu_1(\beta_{12}^2-\mu_1\mu_2)}\bigg]\int_{\mathbb{R}^n} w^4(\sqrt{\lambda}x)\\
&\qquad - 4\beta_{12}t_1t_2\bigg[\frac{\lambda^2\sqrt{(\beta_{12}-\mu_1)(\beta_{12}-\mu_2)}}{\mu_1(\beta_{12}^2-\mu_1\mu_2)}\bigg]\int_{\mathbb{R}^n} w^4(\sqrt{\lambda}x).
\end{aligned}
$$

Therefore, if

$$\beta_{12} > \mu_2 \quad \text{and} \quad \beta_{12}^2 - \mu_1\mu_2 < \beta_{13}(\beta_{12} - \mu_2) + \beta_{23}(\beta_{12} - \mu_1),$$

then, for any $(t_1, 0, t_3) \in \mathbb{R}^3 \setminus (0,0,0)$,

$$E''(U_4)\big[(t_1 w_1, 0, t_3 w_1), (t_1 w_1, 0, t_3 w_1)\big] < 0.$$

And if

$$\beta_{12} > \mu_1 \quad \text{and} \quad \beta_{12}^2 - \mu_1\mu_2 < \beta_{13}(\beta_{12} - \mu_2) + \beta_{23}(\beta_{12} - \mu_1),$$

then, for any $(0, t_2, t_3) \in \mathbb{R}^3 \setminus (0,0,0)$,

$$E''(U_4)\big[(0, t_2 w_1, t_3 w_1), (0, t_2 w_1, t_3 w_1)\big] < 0.$$

In conclusion, the Morse index of U_4 is at least 2 if

$$\beta_{12}^2 > \mu_1\mu_2 \quad \text{and} \quad \beta_{12}^2 - \mu_1\mu_2 < \beta_{13}(\beta_{12} - \mu_2) + \beta_{23}(\beta_{12} - \mu_1).$$

Similarly, if

$$\beta_{13}^2 > \mu_1\mu_3 \quad \text{and} \quad \beta_{13}^2 - \mu_1\mu_3 < \beta_{12}(\beta_{13} - \mu_3) + \beta_{23}(\beta_{13} - \mu_1)$$

and

$$\beta_{23}^2 > \mu_2\mu_3 \quad \text{and} \quad \beta_{23}^2 - \mu_2\mu_3 < \beta_{13}(\beta_{23} - \mu_2) + \beta_{12}(\beta_{23} - \mu_3),$$

we have the conclusion that the Morse indices of U_5, U_6 are at least 2. Since $U_1, \dots, U_6$ are the only six semipositive radial solutions of (2) with only one or two components being zero and since the semipositive radial ground state obtained in [3] has Morse index 1, the semipositive radial ground state is positive. The proof is complete. □

Remark 5. (a) In estimating the lower bound of the Morse index of E at U_1, we have used the three-dimensional subspace spanned by $(w_1, 0, 0), (0, w_1, 0)$ and $(0, 0, w_1)$ as a test subspace, other than subspace which has components be constituted by three of $0, w_1, w_2$ and w_3. The reason is that the subspace $span(w_1, 0, 0), (0, w_1, 0), (0, 0, w_1)$ yields the most transparent estimate among all the subspaces. The same remark applies to the argument of estimating the lower bound of the Morse indices of E at $U_2, \dots, U_6$.

(b) If $\mu_1 \neq \mu_2, \beta_{12} \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ and $\vec{u} = (u_1, u_2, u_3)$ is a semipositive solutions of system (2), then by using Theorem 1.2 of [3], we know that $\vec{u} = (0, 0, 0)$ when $u_3 = 0$. Therefore, in the process of proofing of Theorem 1, we need not to estimate the Morse index of U_4. The similar remarks apply to the cases of $\mu_1 \neq \mu_3$ or $\mu_2 \neq \mu_3$. So we have the following theorem.

Theorem 6. *Assume $\mu_i < \mu_j$, (2) has a positive ground state if $\mu_i \leqslant \beta_{ij} \leqslant \mu_j$,*

$$\beta_{ik} > \max\{\mu_i, \mu_k\}, \qquad \beta_{ik}^2 - \mu_i\mu_k < \beta_{ij}(\beta_{ik} - \mu_k) + \beta_{jk}(\beta_{ik} - \mu_i),$$

and

$$\beta_{jk} > \max\{\mu_j, \mu_k\}, \qquad \beta_{jk}^2 - \mu_j\mu_k < \beta_{ij}(\beta_{jk} - \mu_k) + \beta_{ik}(\beta_{jk} - \mu_j),$$

where $i, j, k = 1, 2$ or 3 and $i, j \neq k$.

Acknowledgment

The author would like to thank Professor Zhaoli Liu for his many useful suggestions in writing this paper and the referee for his suggestions of improving the writing of the paper.

References

[1] N. Akhmediev, A. Ankiewicz, Partially coherent solitons on a finite background, Phys. Rev. Lett. 82 (1999) 2661–2664.
[2] A. Ambrosetti, E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc. 75 (2007) 67–82.
[3] T. Bartsch, Z.-Q. Wang, Note on ground states of nonlinear Schrödinger systems, J. Partial Differ. Equ. 19 (2006) 200–207.
[4] H. Buljan, T. Schwartz, M. Segev, M. Soljacic, D. Christoudoulides, Polychromatic partially spatially incoherent solitons in a noninstantaneous Kerr nonlinear medium, J. Opt. Soc. Amer. B 21 (2004) 397–404.
[5] Jinyong Chang, Zhaoli Liu, Ground states of nonlinear Schrödinger systems, Proc. Amer. Math. Soc. 138 (2010) 687–693.
[6] D.N. Christodoulides, T.H. Coskun, M. Mitchell, M. Segev, Theory of incoherent self-focusing in biased photorefractive media, Phys. Rev. Lett. 78 (1997) 646–649.
[7] B.D. Esry, C.H. Greene, J.P. Burke Jr., J.L. Bohn, Hartree–Fock theory for double condensates, Phys. Rev. Lett. 78 (1997) 3594–3597.
[8] G.M. Genkin, Modification of superfluidity in a resonantly strongly driven Bose–Einstein condensate, Phys. Rev. A 65 (2002) 035604.
[9] F.T. Hioe, Solitary waves for N coupled nonlinear Schrödinger equations, Phys. Rev. Lett. 82 (1999) 1152–1155.
[10] F.T. Hioe, T.S. Salter, Special set and solutions of coupled nonlinear Schrödinger equations, J. Phys. A: Math. Gen. 35 (2002) 8913–8928.

[11] T. Kanna, M. Lakshmanan, Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations, Phys. Rev. Lett. 86 (2001) 5043–5046.
[12] M.K. Kwong, Uniqueness of positive solutions of $-\Delta u + u = u^3$ in $\mathbb{R}^n$, Arch. Ration. Mech. Anal. 105 (1989) 243–266.
[13] T.-C. Lin, J.C. Wei, Ground state of N coupled nonlinear Schrödinger equations in R^n, $n \leqslant 3$, Comm. Math. Phys. 255 (2005) 629–653.
[14] Z.L. Liu, Z.-Q. Wang, Multiple bound states of nonlinear Schrödinger systems, Comm. Math. Phys. 282 (2008) 721–731.
[15] Z.L. Liu, Z.-Q. Wang, Ground states and bound states of a nonlinear Schrödinger system, Adv. Nonlinear Stud. 10 (2010) 175–194.
[16] M. Mitchell, Z. Chen, M. Shih, M. Segev, Self-trapping of partially spatially incoherent light, Phys. Rev. Lett. 77 (1996) 490–493.
[17] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, Comm. Math. Phys. 271 (2007) 199–221.
[18] J.C. Wei, W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, Methods Anal. Appl., in press.